2020

MATHEMATICS — HONOURS

Fifth Paper

(Module - X)

Full Marks : 50

The figures in the margin indicate full marks.

Candidates are required to give their answers in their own words as far as practicable.

Group - A

(Marks : 20)

Answer ***any one*** question.

1. (a) Let V and W be two vector spaces over a field F and $T : V \to W$ be a linear mapping. If Ker$T = \{\theta\}$ and $\{\alpha_1, \alpha_2, ..., \alpha_n\}$ be a basis of V, prove that $\{T(\alpha_1), T(\alpha_2), ...T(\alpha_n)\}$ is a basis of *Im T*.

(b) Let $P_3(\mathbb{R})$ be the real vector space of all polynomials of degree atmost 3. Define $S : P_3(\mathbb{R}) \to P_3(\mathbb{R})$ by $S(p(x)) = p(x + 1)$ for all $p(x) \in P_3(\mathbb{R})$. Find the matrix of S relative to the ordered basis $\{1, x, x^2, x^3\}$ of $P_3(\mathbb{R})$.

(c) If the matrix of a linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^3$ with respect to the ordered basis {(0, 1, 1), (1, 0, 1), (1, 1, 0)} of $\mathbb{R}^3$ is given by $\begin{pmatrix} 0 & 3 & 0 \\ 2 & 3 & -2 \\ 2 & -1 & 2 \end{pmatrix}$, then find dim(*Im T*). 10+6+4

2. (a) Let V and W be two vector spaces of finite dimensions over a field F and let $T : V \to W$ be a linear mapping. Prove that T is invertible if and only if the matrix of T relative to any chosen pair of ordered bases of V and W is nonsingular.

(b) A linear mapping $T : \mathbb{R}^3 \to \mathbb{R}^4$ is defined by $T(x, y, z) = (y + z, z + x, x + y, x + y + z)$, $(x, y, z) \in \mathbb{R}^3$. Find *ImT* and dimension of *ImT*. Also find nullity T. 10+4+4+2

3. (a) If a subgroup H of a group G is defined to be normal if $a\text{H}a^{-} \subseteq \text{H}$ for all $a \in \text{G}$, prove that H is a normal subgroup of G if $a\text{H}a^{-1} = \text{H}$ for all a in G.

(b) Is there any group G of order 6 with the quotient group $G/Z(G)$ of order 3, where $Z(G) = \{a \in G;\ ax = xa \text{ for all } x \in G\}$? Justify your answer.

(c) Prove that any two infinite cyclic groups are isomorphic. Is this true for finite cyclic groups? Justify your answer. 6+4+(6+4)

4. (a) Let G and G' be two groups and $\phi : G \to G'$ be an onto homomorphism. Let $H = \text{Ker}\phi$. Show that $G/H \simeq G'$. Hence show that there does not exist any homomorphism from Z_9 onto Z_6.

(b) Let $a, b \in \mathbb{R}$ and a mapping $T_{ab} : \mathbb{R} \to \mathbb{R}$ be defined by $T_{ab}(x) = ax + b$, $x \in \mathbb{R}$. Let $G = \{T_{ab} : a \neq 0\}$. Assume that $(G, *)$ is a group where $*$ is the composition of mappings. If $H = \{T_{ab} : a = 1\}$, prove that H is a normal subgroup of $(G, *)$. (6+4)+10

Group - B

(Marks : 15)

Answer ***any one*** question.

5. If (A_1, A_2) is a covariant vector in Cartesian coordinates x^1, x^2 where $A_1 = \dfrac{x^1}{x^2}$ and $A_2 = \dfrac{x^2}{x^1}$, find its components in polar coordinates. 15

6. (a) Show that angle between two vectors at a point in a Riemannian space is an invariant under coordinate transformation.

(b) Prove that the covariant derivative of the metric tensor g_{ij} vanishes. 15

7. Prove that if A_i and B_i be two covariant vectors, then $A_iB_j - A_jB_i$ is a skew symmetric tensor. 15

8. Calculate the quantities (g^{ij}) and (g_{ij}) where the metric is given by $ds^2 = (dx_1)^2 + {x_1}^2(dx_2)^2 + {x_1}^2 \sin^2 x_2 (dx_3)^2$. Also calculate $\left\{ \begin{matrix} & 2 & \\ 2 & & 1 \end{matrix} \right\}$. 15

9. Prove that $A^{jk}\left[ij, k\right] = \dfrac{1}{2} A^{jk} \dfrac{\partial g_{jk}}{\partial x^i}$, where A^{ij} are components of a symmetric contravariant tensor of rank 2. 15

Answer either **Group - C** *or* **Group - D**

Group - C

(Marks : 15)

Answer ***any one*** question.

10. (a) (i) Show that $L(t^n) = \dfrac{n!}{p^{n+1}}$, where n is a positive integer and $t > 0$.

(ii) Find the Laplace transform of $f(t)$ defined by

$$f(t) = \begin{cases} 0 & \text{if } 0 < t < 2 \\ 3 & \text{if } t > 2 \end{cases}$$

(b) Using Laplace transform solve $\frac{d^2y}{dx^2}+6\frac{dy}{dx}+10y=0,$ when $y(0) = 1,\ y\left(\frac{\pi}{4}\right)=\sqrt{2}.$

(c) Find the power series solution of $\frac{d^2y}{dx^2}+(x-3)\frac{dy}{dx}+y=0$ near $x = 2$. 15

11. (a) Evaluate $L^{-1}\left(\frac{1}{(p-3)^2}\cdot\frac{1}{p+4}\right)$

(b) Using Laplace transform, solve $y''(t) + 4y'(t) + 4y(t) = 4e^{-2t}$, $y(0) = -1$ and $y'(0) = 4$.

(c) State a set of sufficient conditions for the existence of Laplace transform. Show that Laplace transform is a linear operator. If f is the Laplace transform of F, determine the Laplace transform of G defined by $G(t)=\begin{cases}0 & 0<t<a\\ F(t-a) & t>a\end{cases}$ 15

Group - D

(Marks : 15)

Answer ***any two*** questions.

12. State and prove the necessary and sufficient condition for a connected graph to be an Euler graph. 7½

13. If n, e and f are respectively the number of vertices, number of edges and number of faces of a planar graph, then show that $n-e+f=2$. 7½

14. If $\delta(G)$ and $\Delta(G)$ denote the minimum and maximum degrees of the vertices of a graph with n vertices and e edges then prove that $\delta(G)\leqslant\frac{2e}{n}\leqslant\Delta(G)$. 7½

15. (a) Define complement of a graph. Show that the complement of P_4(a path on 4 vertices) is again P_4.

(b) In a tree T prove that there exists one and only one path between every pair of vertices. 7½

16. Construct the minimum spanning tree for the given graph using Prim's Algorithm. 7½